Inequalities (2) and (3) imply the required result.
In order to make this solution self-contained, the definition of majorizing and the Majorizing Inequality are explained here.
The explanations are excerpted from a nice short article by Murray S. Klamkin (1921-2004) who was one of the greatest problems composer.
M. S. Klamkin, *On a "Problem of the Month"*, Crux Mathematicorum, Volume 28, Number 2, page 86, 2002.
"If A and B are vectors $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n)$ where $a_1 \geq a_2 \geq \ldots \geq a_n$, $b_1 \geq b_2 \geq \ldots \geq b_n$, and $a_1 \geq b_1,\ a_1 + a_2 \geq b_1 + b_2$, $a_1 + a_2 + \cdots + a_{n-1} \geq b_1 + b_2 + \cdots + b_{n-1}$, $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$, we say that A majorizes B and write it as $A \succ B$. Then, if F is a convex function,

$$F(a_1) + F(a_2) + \cdots + F(a_n) \geq F(b_1) + F(b_2) + \cdots + F(b_n)."$$

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Soumava Chakraborty, Kolkata, India; Pedro Acosta De Leon, Massachusetts Institute of Technology Cambridge, MA; Bruno Salgueiro Fanego, Viveiro, Spain. Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Athens, Greece, and the proposers.

5473: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $x_1, \cdots, x_n$ be positive real numbers. Prove that for $n \geq 2$, the following inequality holds:

$$\left(\sum_{k=1}^{n} \frac{\sin x_k}{((n-1)x_k + x_{k+1})^{1/2}}\right)\left(\sum_{k=1}^{n} \frac{\cos x_k}{((n-1)x_k + x_{k+1})^{1/2}}\right) \leq \frac{1}{2}\sum_{k=1}^{n} \frac{1}{x_k}.$$

(Here the subscripts are taken modulo n.)

Solution 1 by Moti Levy, Rehovot, Israel

The following three facts will be used in this solution:
1)

$$\left(\sum_{k=1}^{n} a_k \sin x_k\right)\left(\sum_{k=1}^{n} a_k \cos x_k\right) \leq \frac{1}{2}\left(\sum_{k=1}^{n} a_k\right)^2. \tag{4}$$

This can be shown by expanding the left hand side and using the facts that $\sin x_k \cos x_k \leq \frac{1}{2}$ and $\sin x_j \cos x_k + \cos x_j \sin x_k \leq 1$.
2)

$$\left(\sum_{k=1}^{n} \frac{\sqrt{a_k}}{n}\right)^2 \leq \sum_{k=1}^{n} \frac{a_k}{n}. \tag{5}$$

This is implied from $M_{\frac{1}{2}} \leq M_1$ where M_k are power means.
3)

$$\frac{1}{px + qy} \leq \frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right), \quad p, q \geq 0 \ \ \textit{and} \ \ p + q = 1. \tag{6}$$

This can be shown by Jensen's inequality.

Now let

$$a_k := \frac{1}{((n-1)\,x_k + x_{k+1})^{\frac{1}{2}}}.$$

Then

$$\begin{aligned} LHS &:= \left(\sum_{k=1}^{n} \frac{\sin x_k}{((n-1)\,x_k + x_{k+1})^{\frac{1}{2}}}\right)\left(\sum_{k=1}^{n} \frac{\cos x_k}{((n-1)\,x_k + x_{k+1})^{\frac{1}{2}}}\right) \\ &= \left(\sum_{k=1}^{n} a_k \sin x_k\right)\left(\sum_{k=1}^{n} a_k \cos x_k\right) \le \frac{1}{2}\left(\sum_{k=1}^{n} a_k\right)^2. \end{aligned}$$

By (5),

$$\begin{aligned} LHS &\le \frac{1}{2}\left(\sum_{k=1}^{n} a_k\right)^2 \le \frac{n}{2}\sum_{k=1}^{n} a_k = \frac{n}{2}\sum_{k=1}^{n} \frac{1}{(n-1)\,x_k + x_{k+1}} \\ &= \frac{1}{2}\sum_{k=1}^{n} \frac{1}{\frac{n-1}{n}x_k + \frac{1}{n}x_{k+1}}. \end{aligned}$$

Set $p = \frac{n-1}{n}$ and $q = \frac{1}{n}$, then by (6)

$$\frac{1}{2}\sum_{k=1}^{n} \frac{1}{\frac{n-1}{n}x_k + \frac{1}{n}x_{k+1}} \le \frac{1}{4}\sum_{k=1}^{n}\left(\frac{1}{x_k} + \frac{1}{x_{k+1}}\right) = \frac{1}{2}\sum_{k=1}^{n}\frac{1}{x_k}.$$

Solution to 2 by Kee-Wai Lau, Hong Kong, China

Since $2ab \le a^2 + b^2$ for any real numbers a and b, so by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & 2\left(\sum_{k=1}^{n} \frac{\sin x_k}{((n-1)x_k + x_{k+1})^{1/2}}\right)\left(\sum_{k=1}^{n} \frac{\cos x_k}{((n-1)x_k + x_{k+1})^{1/2}}\right) \\ \le\ & \left(\sum_{k=1}^{n} \frac{\sin x_k}{((n-1)x_k + x_{k+1})^{1/2}}\right)^2 + \left(\sum_{k=1}^{n} \frac{\cos x_k}{((n-1)x_k + x_{k+1})^{1/2}}\right)^2 \\ \le\ & \left(n\sum_{k=1}^{n} \frac{\sin^2 x_k}{(n-1)x_k + x_{k+1}} + \sum_{k=1}^{n} \frac{\cos^2 x_k}{(n-1)x_k + x_{k+1}}\right) \\ =\ & n\sum_{k=1}^{n} \frac{1}{(n-1)x_k + x_{k+1}}. \end{aligned}$$

Applying Jensen's inequality to the convex function $\frac{1}{x}$ for $x > 0$, we have

$$\frac{n-1}{x_k} + \frac{1}{x_{k+1}} \ge n\left(\frac{1}{\frac{(n-1)x_k + x_{k+1}}{n}}\right) = \frac{n^2}{(n-1)x_k + x_{k+1}}.$$

It follows that $n\sum_{k=1}^{n}\frac{1}{(n-1)x_k+x_{k+1}} \leq \frac{1}{n}\left(\sum_{k=1}^{n}\frac{n-1}{x_k}+\sum_{k=1}^{n}\frac{1}{x_{k+1}}\right)=\sum_{k=1}^{n}\frac{1}{x}.$

Thus the inequality of the problem holds.

Solution 3 by Arkady Alt , San Jose, CA

By Cauchy Inequality $\sum_{k=1}^{n}\frac{\sin x_k}{((n-1)\,x_k+x_{k+1})^{1/2}} \leq \sqrt{\sum_{k=1}^{n}\frac{1}{(n-1)\,x_k+x_{k+1}}}\cdot\sqrt{\sum_{k=1}^{n}\sin^2 x_k}$

and $\sum_{k=1}^{n}\frac{\cos x_k}{((n-1)\,x_k+x_{k+1})^{1/2}} \leq \sqrt{\sum_{k=1}^{n}\frac{1}{(n-1)\,x_k+x_{k+1}}}\cdot\sqrt{\sum_{k=1}^{n}\cos^2 x_k}.$

Also by AM-GM inequality

$$\sqrt{\sum_{k=1}^{n}\sin^2 x_k}\cdot\sqrt{\sum_{k=1}^{n}\cos^2 x_k} \leq \frac{1}{2}\left(\sum_{k=1}^{n}\sin^2 x_k+\sum_{k=1}^{n}\cos^2 x_k\right)=\frac{1}{2}\sum_{k=1}^{n}\left(\sin^2 x_k+\cos^2 x_k\right)=\frac{n}{2}.$$

Thus, $\left(\sum_{k=1}^{n}\frac{\sin x_k}{((n-1)\,x_k+x_{k+1})^{1/2}}\right)\left(\sum_{k=1}^{n}\frac{\cos x_k}{((n-1)\,x_k+x_{k+1})^{1/2}}\right) \leq \frac{n}{2}\sum_{k=1}^{n}\frac{1}{(n-1)\,x_k+x_{k+1}}$

and it remains to prove the inequality

$$\frac{n}{2}\sum_{k=1}^{n}\frac{1}{(n-1)\,x_k+x_{k+1}} \leq \frac{1}{2}\sum_{k=1}^{n}\frac{1}{x_k} \iff \sum_{k=1}^{n}\frac{1}{(n-1)\,x_k+x_{k+1}} \leq \frac{1}{n}\sum_{k=1}^{n}\frac{1}{x_k}.$$

By the Cauchy Inequality

$$((n-1)\,x_k+x_{k+1})\left(\frac{n-1}{x_k}+\frac{1}{x_{k+1}}\right) \geq n^2 \iff \frac{1}{(n-1)\,x_k+x_{k+1}} \leq \frac{1}{n^2}\left(\frac{n-1}{x_k}+\frac{1}{x_{k+1}}\right)$$

then $\sum_{k=1}^{n}\frac{1}{(n-1)\,x_k+x_{k+1}} \leq \frac{1}{n^2}\sum_{k=1}^{n}\left(\frac{n-1}{x_k}+\frac{1}{x_{k+1}}\right)=\frac{1}{n}\sum_{k=1}^{n}\frac{1}{x_k}.$

Also solved by Ed Gray, Highland Beach, FL; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5474: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a, b \in \Re, b \neq 0$. Calculate

$$\lim_{n\to\infty}\begin{pmatrix} 1-\frac{a}{n^2} & \frac{b}{n} \\ \frac{b}{n} & 1+\frac{a}{n^2}. \end{pmatrix}^n.$$

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain